Appendix A: Notes on Energy Methods in linear elasticity

Supplementary material for the course of Solid Mechanics

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APPENDIX A: Energy concepts and analysis in linear elasticity

In what follows, we are concerned with the equilibrium of a solid, made of an isotropic homogeneous linearly elastic material and subjected to body forces, prescribed displacements and/or tractions on its boundary.

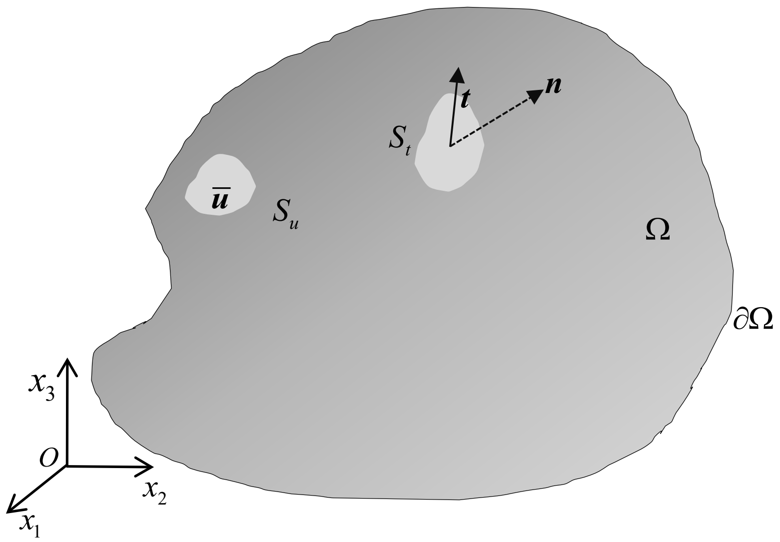

Fig. A1: Schematic of a solid in equilibrium subjected to boundary tractions and displacement.

We will use simple arguments to prove the uniqueness of an elastic solution when it exists. We will also discuss methods of solution based on energy and define certain approximate methods of solution. For completeness, we start with the system of equations for a solid in static equilibrium.

Boundary value problems of static linear elasticity.

For a solid in equilibrium, we have the following system of equations (Chapter 7: Botsis and Deville, 2018).

The Field Equations of Linear Elastostatics

The system of field equations consists of:

- the 3 *equations of equilibrium*

  $$\sigma_{ij,j} + f_i = 0 \quad , \quad div\boldsymbol{\sigma} + \boldsymbol{f} = 0 \qquad \text{(A.1)}$$

  where $\boldsymbol{f}$ is body force vector.

- the 6 equations defining the *strain-displacement relation*

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad , \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) \tag{A.2}$$

- the 6 equations defining the *isotropic homogeneous stress-strain relation*

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad , \quad \boldsymbol{\sigma} = \lambda tr\boldsymbol{\varepsilon}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \tag{A.3a}$$

Here the Lamé constants λ and μ, are independent of $\boldsymbol{x}$. They are related to Young's modulus E and Poisson's ratio ν by,

$$\lambda = E\nu/(1+\nu)(1-2\nu), \quad \mu = E/2(1+\nu). \tag{A.3b}$$

A simple counting shows that we have 15 unknowns (3 displacement components u_i, 6 strain components ε_{ij} and 6 stress components σ_{ij}) and 15 equations. Accordingly, the problem is well - posed.

There are two ways for combining the 15 equations of (A.1)-(A.3). The first way corresponds to taking the three displacement components u_i as the unknowns. Consequently, we introduce (A.2) into (A.3) to obtain,

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{A.4}$$

Substituting (A.4) into (A.1) gives,

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0\,. \tag{A.5a}$$

These are the well-known *Navier equations*. Using the usual differential operator notations, we can write (A.5a) as,

$$(\lambda + \mu)\nabla(div\boldsymbol{u}) + \mu\Delta\boldsymbol{u} + \boldsymbol{f} = 0 \tag{A.5b}$$

where Δ represents the Laplace operator. For (A.5) to make sense, the displacement must be twice continuously differentiable.

The second way consists in considering the 6 stress components σ_{ij} as the unknowns. To proceed, we first replace the stress-strain relation (A.3) by its inverse, the *strain-stress relation*,

$$\boldsymbol{\varepsilon} = -\frac{\lambda}{2\mu(3\boldsymbol{\lambda} + 2\mu)} tr\boldsymbol{\sigma}\boldsymbol{I} + \frac{1}{2\mu}\boldsymbol{\sigma} = -\frac{\nu}{E} tr\boldsymbol{\sigma}\boldsymbol{I} + \frac{1+\nu}{E}\boldsymbol{\sigma} \tag{A.6a}$$

$$\varepsilon_{ij} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} \tag{A.6b}$$

Next, substituting (A.6b) into the 6 *strain compatibility equations* (Chapter 2: Botsis and Deville, 2018),

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0 \tag{A.7}$$

we obtain

$$(1+\nu)\sigma_{ij,kk} - \nu\sigma_{mm,nn}\delta_{ij} + \sigma_{pp,ij} - (1+\nu)(\sigma_{iq,qj} + \sigma_{jr,ri}) = 0 \tag{A.8}$$

Taking the derivatives of the equilibrium equations (A.1) twice, we have the following result,

$$\sigma_{iq,qi} + \sigma_{jr,ri} = -f_{i,j} - f_{j,i} \tag{A.9}$$

Using (A.9), relation (A.8) becomes,

$$(1+\nu)\sigma_{ij,kk} - \nu\sigma_{mm,nn}\delta_{ij} + \sigma_{pp,ij} + (1+\nu)(f_{i,j} + f_{j,i}) = 0 \tag{A.10}$$

Taking the trace of this equation results in,

$$(1-\nu)\sigma_{mm,nn} = -(1+\nu)f_{k,k} \tag{A.11}$$

Using this relation in (A.10) and assuming $\nu \neq -1$, we obtain,

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu}f_{n,n}\delta_{ij} = 0 \tag{A.12}$$

These are *the stress compatibility equations of Beltrami-Michell*. If body forces are constant, (A.12) reduces to,

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} = 0 \tag{A.13}$$

Note that (A.13) are trivially satisfied when σ_{ij} are affine functions of $\boldsymbol{x}$.

The Boundary Value Problems of Linear Elastostatics. The preceding field equations can be solved only when appropriate boundary conditions are imposed. Consider a solid occupying a domain Ω of $\mathbb{R}^3$ with $\partial\Omega$ as its boundary (Figure A1). In a general manner, we partition the boundary surface $\partial\Omega$ into two parts: $\partial\Omega = S_u \cup S_t$ with $S_u \cap S_t = \varnothing$ where:

S_u represents the part of $\partial\Omega$ on which the displacement $\overline{u}_i$ is prescribed, i.e.

$$u_i = \overline{u}_i \quad \text{on} \quad S_u \tag{A.14}$$

S_t denotes the part of $\partial\Omega$ on which the surface traction $\overline{t}_i$ are assigned, i.e.

$$t_i = \sigma_{ij} n_j = \overline{t}_i \quad \text{on} \quad S_t \tag{A.15}$$

where n_j are the components of an outward unit normal to S_t.

If neither S_u nor S_t is empty, the corresponding boundary condition is called a *mixed boundary condition*. When S_t is empty, we are concerned with a *displacement boundary condition*. In the case of $S_u = \varnothing$, the boundary condition corresponds to a *traction boundary condition*.

With the preceding definitions, we can formulate the classical boundary value problems of linear elastostatics.

The mixed boundary value problem in terms of displacement components. Condition (A.15) can be expressed in terms of u_i via (A.4),

$$\lambda u_{k,k} n_i + \mu(u_{i,j} + u_{j,i}) n_j = \overline{t}_i \quad \text{on} \quad S_t \ . \tag{A.16}$$

Navier's equations (A.5) together (A.14) and (A.16) give the *formulation of the mixed problem in terms of displacement components*,

$$(\lambda + \mu) u_{k,ki} + \mu u_{i,mm} + f_i = 0 \quad \text{over} \quad \Omega \tag{A.17a}$$

$$u_i = \overline{u}_i \quad \text{on} \quad S_u \tag{A.17b}$$

$$\lambda u_{k,k} n_i + \mu(u_{i,j} + u_{j,i}) n_j = \overline{t}_i \quad \text{on} \quad \partial\Omega \quad . \tag{A.17c}$$

The displacement boundary value problem. If $S_t = \varnothing$ so that $S_u = \partial\Omega$, the foregoing formulation reduces to,

$$(\lambda + \mu) u_{k,ki} + \mu u_{i,mm} + f_i = 0 \quad \text{over} \quad \Omega \tag{A.18a}$$

$$u_i = \overline{u}_i \qquad \text{on} \quad \partial\Omega \ . \tag{A.18b}$$

The traction boundary value problem in terms of displacement components. If $S_u = \varnothing$, then $S_t = \partial\Omega$. We can simplify (A.17) to,

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,mm} + f_i = 0 \quad \text{over} \ \ \Omega, \tag{A.19a}$$

$$\lambda u_{k,k} n_i + \mu(u_{i,j} + u_{j,i})n_j = \overline{t_i} \quad \text{on} \ \ S_t. \tag{A.19b}$$

The traction boundary value problem in terms of stress components. We set $S_u = \varnothing$ so that $S_t = \partial\Omega$. The equilibrium equations (A.1), the stress compatibility equations (A.12) and the condition (A.15) constitute the *formulation of the traction problem in terms of stress components*,

$$\sigma_{ij,j} + f_i = 0 \quad \text{over} \ \ \Omega \tag{A.20a}$$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu} f_{n,n}\delta_{ij} = 0 \quad \text{over} \quad \Omega \tag{A.20b}$$

$$t_i = \sigma_{ij} n_j = \overline{t_i} \quad \text{on} \quad S_t. \tag{A.20c}$$

It is worth noticing that the traction boundary value problem can be formulated in terms of displacement components whereas the mixed or displacement boundary value problem cannot generally be formulated in terms of stress components.

Energy principles

After the boundary value problem is formulated, two important questions need to be asked: (a) does a solution exist? (b) if it exists, is it possible to have more than one? Although a demonstration of the existence of solution is beyond the scope of this introduction, we do examine here its uniqueness.

In solving problems of linear elasticity, we are often led to postulate certain forms of displacement and stress fields and verify if equations (A.1) - (A.3) together with prescribed boundary conditions are satisfied. If the answer is positive and if the uniqueness of solution can be ensured, we can conclude that the postulated fields constitute the solution of the problem.

In developing the theory and models in mechanics, solids and structures, the energy of the system plays an indispensable and significant role. Besides, in several cases in engineering, we encounter difficulties in obtaining exact solutions of the governing equations. Thus, it is important to consider various approximate methods. A group of methods is based on results that these equations can be obtained from minimization of the system's potential energy.

Below, present some important concepts on energy give a general result about the uniqueness of solution in the isotropic case. Its proof is based on an energy equation and can easily be extended to the anisotropic case. Afterwards, we discuss the principles of virtual work and potential energy and their use in mechanics. These principles are very useful in the solution of important problems in engineering. Application examples are given at the end.

Theorem of work and energy. Let $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ be the displacement, strain and stress fields satisfying the equations (A.1) - (A.3) together with the boundary conditions (A.14) and (A.15). By taking the scalar product of (A.1) with u_i, and integrating over Ω, we can write,

$$\int_\Omega \sigma_{ij,j} u_i dv + \int_\Omega f_i u_i dv = 0 . \qquad \text{(A.21)}$$

Using the divergence theorem, the symmetry $\sigma_{ij} = \sigma_{ji}$ and Cauchy's formula, $t_i = \sigma_{ij} n_j$ we can modify the first term as follows,

$$\begin{aligned}
\int_\Omega \sigma_{ij,j} u_i dv &= \int_\Omega \left((\sigma_{ij} u_i)_{,j} - \sigma_{ij} u_{i,j} \right) dv = \int_\Omega \left((\sigma_{ij} u_i)_{,j} \right) dv - \int_\Omega \left(\sigma_{ij} u_{i,j} \right) dv \\
&= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left(\frac{1}{2} (\sigma_{ij} u_{i,j} + \sigma_{ij} u_{i,j}) \right) dv \\
&= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left(\frac{1}{2} (\sigma_{ij} u_{i,j} + \sigma_{ji} u_{j,i}) \right) dv \\
&= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left(\frac{1}{2} (\sigma_{ij} u_{i,j} + \sigma_{ij} u_{j,i}) \right) dv \qquad \text{(A.22)} \\
&= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left(\sigma_{ij} \frac{1}{2} (u_{i,j} + u_{j,i}) \right) dv \\
&= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \sigma_{ij} \varepsilon_{ij} dv \\
&= \int_{\partial\Omega} t_i u_i ds - \int_\Omega \sigma_{ij} \varepsilon_{ij} dv.
\end{aligned}$$

Substitution of (A.22) into (A.21) results in,

$$\int_\Omega \sigma_{ij} \varepsilon_{ij} dv = \int_{\partial\Omega} t_i u_i ds + \int_\Omega f_i u_i dv . \qquad \text{(A.23)}$$

Observe that the left-hand side of this expression is the work done by the stress $\boldsymbol{\sigma}$ on the strain $\boldsymbol{\varepsilon}$, that corresponds to the change in internal energy, while the right-hand side represents the sum of work done by the surface and body forces during the displacement $\boldsymbol{u}$. Since the stress components σ_{ij} are related to the strain components ε_{ij} by (A.3), we have,

$$\int_{\Omega} \sigma_{ij}\varepsilon_{ij} dv = \int_{\Omega} \left(\lambda \varepsilon_{ii}\varepsilon_{kk} + 2\mu \varepsilon_{ij}\varepsilon_{ij} \right) dv = 2\int_{\Omega} W(\varepsilon_{ij}) dv . \qquad \text{(A.24)}$$

where use has been made of the strain energy density $W(\varepsilon_{ij})$ for an isotropic linearly elastic material. In view of (A.24), (A.23) can be written as,

$$\frac{1}{2}\left(\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv \right) = \int_{\Omega} W(\varepsilon_{ij}) dv . \qquad \text{(A.25)}$$

In the literature, (A.25) is referred to as the *theorem of work and energy,* or principle of conservation of mechanical energy, for an isotropic linearly elastic solid. It states that half of work of the external forces, applied to the solid, equals the strain energy of deformation.

Potential energy. At this point, we introduce an important energy function in mechanics, defined as the *potential energy* Π, and given by the difference of the strain energy U and the work of applied forces $\mathbb{W}$,

$$\Pi = U - \mathbb{W} . \qquad \text{(A.26a)}$$

For a linear elastic solid, they are expressed as follows,

$$\mathbb{W} = \int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv \qquad \text{(A.26b)}$$

$$U = \int_{\Omega} \frac{1}{2} \sigma_{ij}\varepsilon_{ij} dv = \int_{\Omega} \frac{1}{2}\left(\lambda \varepsilon_{kk}\delta_{ij} + 2\mu \varepsilon_{ij} \right)\varepsilon_{ij} dv = \int_{\Omega} W(\varepsilon_{ij}) dv . \qquad \text{(A.26c)}$$

With these expressions the potential energy for a linear elastic solid is given by,

$$\Pi = U - \mathbb{W} = \int_{\Omega} W(\varepsilon_{ij}) dv - \left(\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv \right). \qquad \text{(A.27d)}$$

Accounting for (A.25), we obtain,

$$\Pi = U - \mathbb{W} = \int_{\Omega} W(\varepsilon_{ij}) dv - \left(\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv \right) = -\int_{\Omega} W(\varepsilon_{ij}) dv . \qquad \text{(A.27e)}$$

The potential energy is taken as zero when the body is in its undeformed state. In addition, it is important to note that $\mathbb{W}$ is not the work done by the external forces in deforming the body between loading and equilibrium. The work done by the external forces is $\frac{1}{2}\mathbb{W}$. Thus, the potential energy can be interpreted as energy ‘recovered upon unloading’, or energy for processes other than deformation, and for a linear elastic solid it is,

$$\Pi = U - \mathbb{W} = -U = -\frac{1}{2}\mathbb{W} \tag{A.27f}$$

because $U = \frac{1}{2}\mathbb{W}$ as shown in (A.25).

Volumetric and distortional components of the strain energy. It is very useful to express the elastic strain density in terms of volumetric $W_p(\varepsilon_{ij})$ and distortional $W_d(\varepsilon_{ij})$ parts. To do so, we introduce the strain deviator ε_{ij}^d,

$$\varepsilon_{ij}^d = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\,. \tag{A.28a}$$

Thus, it is not difficult to show that the strain energy is,

$$W(\varepsilon_{ij}) = \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{9}{2}K\left(\varepsilon_0\right)^2 + \mu\varepsilon_{ij}^d\varepsilon_{ij}^d = W_p(\varepsilon_{ij}) + W_d(\varepsilon_{ij})\,. \tag{A.28b}$$

Here $K = (3\lambda + 2\mu)/3$ is the *bulk modulus* and $\varepsilon_0 = \frac{1}{3}\varepsilon_{ii}$ .With this expression, (A.25) can be written as,

$$\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv = \int_{\Omega}\left(9K\left(\varepsilon_0\right)^2 + 2\mu\varepsilon_{ij}^d\varepsilon_{ij}^d\right)dv\,. \tag{A.29}$$

For an isotropic material, the stability hypothesis amounts to the condition that,

$$K > 0, \quad \mu > 0\,. \tag{A.30}$$

These inequalities are physically meaningful because they cannot be zero or negative.

Uniqueness of the solution. After the foregoing introduction, we can state and demonstrate the uniqueness of the solution with the following theorem.

Theorem. Let $\left(\boldsymbol{u}^{(1)},\boldsymbol{\varepsilon}^{(1)},\boldsymbol{\sigma}^{(1)}\right)$ and $\left(\boldsymbol{u}^{(2)},\boldsymbol{\varepsilon}^{(2)},\boldsymbol{\sigma}^{(2)}\right)$ be two sets of displacement, strain and stress fields satisfying the field equations (A.1) - (A.3) together with the boundary conditions (A.14) - (A.15).

If $S_u \neq \varnothing$, then

$$\boldsymbol{u}^{(2)} = \boldsymbol{u}^{(1)}, \quad \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(1)}, \quad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)}.$$

If $S_u = \varnothing$, then

$$\boldsymbol{u}^{(2)} = \boldsymbol{u}^{(1)} + \boldsymbol{w}, \quad \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(1)}, \quad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)}$$

where $\boldsymbol{w}$ is an infinitesimal rigid body rotation.

Demonstration

Let's define the following differences $\boldsymbol{u} = \boldsymbol{u}^{(2)} - \boldsymbol{u}^{(1)}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(2)} - \boldsymbol{\varepsilon}^{(1)}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{(2)} - \boldsymbol{\sigma}^{(1)}$.

Then $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ fulfill the field equations (A.1) - (A.3) with zero body forces, i.e.

$$f_i = 0, \tag{a}$$

and $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ satisfy the following boundary conditions:

$$u_i = 0 \;\; on \;\; S_u \quad , \quad t_i = \sigma_{ij} n_j = 0 \;\; on \;\; S_t \tag{b}$$

Thus, (A.29) becomes,

$$\int_\Omega \left(K\varepsilon_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ij}^d \varepsilon_{ij}^d \right) dv = 0. \tag{c}$$

Since $K > 0$ and $\mu > 0$, (c) holds if and only if $\varepsilon_{ii} = 0$ and $\varepsilon_{ij}^d = 0$. This amounts to writing,

$$\varepsilon_{ij} = \varepsilon_{ij}^d = 0 \tag{d}$$

which, due to (A.3a), implies,

$$\sigma_{ij} = 0. \tag{e}$$

Moreover, (d) means that $\boldsymbol{u}$ is at most an infinitesimal rigid displacement $\boldsymbol{w}$. If $S_u \neq \varnothing$, $\boldsymbol{u} = \boldsymbol{0}$ on S_u, so that we must have $\boldsymbol{w} = \boldsymbol{0}$ everywhere. However, if $S_u = \varnothing$, i.e., if surface tractions are assigned over the entire boundary $\partial\Omega$, $\boldsymbol{u}^{(2)}$ and $\boldsymbol{u}^{(1)}$ may effectively differ by an infinitesimal rigid displacement $\boldsymbol{w}$.

From the above demonstration, we see that the key condition ensuring the uniqueness of solution are the inequalities (or stability condition) (A.30). If this condition holds, the theorem tells us that:

- *for a mixed or displacement boundary value problem, the displacement, strain and stress fields are unique.*
- *for a traction boundary value problem, the strain and stress fields are unique while the displacement field is unique only to within an infinitesimal rigid displacement.*

Principle of Virtual Work. The theory of linear elasticity may be developed from energy considerations. This alternative approach has the advantage of obtaining approximate solutions of several important problems in solids and structures. In this section, we present the principle of *virtual work* and the principle of *minimum potential energy* in linear elasticity, which are also the theoretical underpinnings for a variety of numerical methods, such as *finite elements*, currently used to solve boundary value problems of linear elasticity.

Consider a solid in equilibrium under the action of body forces, subjected to mixed boundary conditions,

$$\sigma_{ij,j} + f_i = 0 \quad over \quad \Omega \tag{A.31a}$$

$$u_i = \overline{u}_i \quad on \quad S_u \tag{A.31b}$$

$$\sigma_{ij} n_j = \overline{t}_i \quad on \quad S_t \tag{A.31c}$$

If we need to move the solid to another position, additional force is required and thus, the original system of forces must be altered in such a motion. Let’s introduce the notion of a virtual displacement field. The word "virtual" means not necessarily real. It is an arbitrary displacement which does not affect the force system acting on the solid and during the process of its application, all forces remain constant in magnitude and direction.

The magnitude of a virtual displacement is arbitrary. However, because in an infinitesimal actual displacement, the resulting changes in the acting forces are small and considered negligible, in comparison to the forces themselves, a virtual displacement is often considered as an infinitesimal real displacement and thus, the changes in the forces are negligible.

In what follows, a virtual displacement is defined as a possible variation $\delta \boldsymbol{u}$ of the real displacement field $\boldsymbol{u}$ of the solid. It is arbitrary and is subjected to the following restrictions:

1. The components of $\delta \boldsymbol{u}$ $(\delta u_1, \delta u_2, \delta u_3)$ are small, continuous and single-valued functions.

2. During a virtual displacement, surface, body forces and internal stresses are constant in magnitude and direction.

3. The virtual displacement does not violate the prescribed displacement condition (A.31b). This definition implies that $\delta \boldsymbol{u}$ is a virtual displacement if and only if,

$$\delta u_i = 0 \quad on \quad S_u \tag{A.32a}$$

4. With each virtual displacement vector $\delta \boldsymbol{u}$, we associate a virtual strain tensor $\delta \boldsymbol{\varepsilon}$,

$$\delta\varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \tag{A.33a}$$

$$\delta\varepsilon_{ij} = \frac{1}{2}\delta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(\delta u_i) + \frac{\partial}{\partial x_i}(\delta u_j)\right) \tag{A.33b}$$

It is important to note here that the operators d and δ can be interchanged as seen in (A.33b).

Based on the preceding discussion, we can state the following important theorem.

Theorem of Virtual Work. Let $\delta \boldsymbol{u}$ be a virtual displacement field and let $\boldsymbol{\sigma}$ be a stress field verifying the equilibrium equation (A.31a) and the traction boundary condition (A.31c). Then, the virtual strain energy is equal to the virtual work done by the applied forces,

$$\delta U = \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}dv = \int_{S_t} \bar{t}_i \delta u_i ds + \int_\Omega f_i \delta u_i dv \tag{A.34a}$$

Demonstration

By means of a change in notation, the arguments we use here are identical to those leading to formula (A.23). That is, we take the scalar product of (A.31a) with δu_i and integrate over Ω to obtain,

$$\int_\Omega \sigma_{ij,j}\delta u_i dv + \int_\Omega f_i \delta u_i dv = 0 \tag{A.34b}$$

Using the divergence theorem and the symmetry $\sigma_{ij} = \sigma_{ji}$, we can write,

$$\int_\Omega \sigma_{ij,j}\delta u_i dv = \int_\Omega \left((\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij}\delta u_{i,j}\right)dv = \int_{\partial\Omega} \sigma_{ij} n_j \delta u_i ds - \int_\Omega \sigma_{ij}\delta\varepsilon_{ij} dv \tag{A.34c}$$

Substituting this into (A.34b) we obtain,

$$\int_{\Omega} \sigma_{ij} \delta\varepsilon_{ij} dv = \int_{\partial\Omega} \sigma_{ij} n_j \delta u_i ds + \int_{\Omega} f_i \delta u_i dv \tag{A.34d}$$

Due to (A.31c) and (A.32), (A.34d) becomes (A.34a).The two integrals on the right hand side give the virtual work of the applied forces,

$$\delta \mathbb{W} = \int_{\partial\Omega} \sigma_{ij} n_j \delta u_i ds + \int_{\Omega} f_i \delta u_i dv \tag{A.34e}$$

In mechanics, (A.34a) is the result of the *principle of virtual work*. It is often used to establish the governing equation of a mechanical system. It is also important to remark that in the demonstration of the virtual work theorem, no constitutive relations have been involved and thus, (A.34a) holds regardless of material properties.

Principle of Minimum Potential Energy. We can define now a very important principle in mechanics. Using (A.34a and e) we can write,

$$\delta\Pi = \delta\left(U - \mathbb{W}\right) = \int_{\Omega} \sigma_{ij} \delta\varepsilon_{ij} dv - \left(\int_{\partial\Omega} \sigma_{ij} n_j \delta u_i ds + \int_{\Omega} f_i \delta u_i dv\right) = 0 . \tag{A.35}$$

This last equation implies that at equilibrium, the potential energy of a solid takes a stationary value. This leads to the definition of the principle of potential energy as follows: *of all displacement fields satisfying the continuity and boundary conditions, of the solid in equilibrium, the actual displacement field makes the potential energy a stationary value*. It can also be shown that this stationary value is a minimum.

Now we proceed to show that the solution of the mixed boundary value problem (A.17), if it exists, can be characterized by the displacement field minimizing the potential energy Π. Before proceeding, we need to define a *kinematically admissible displacement*.

A displacement field $\tilde{\boldsymbol{u}}$ is said to be *kinematically admissible* if it respects the assigned displacement on S_u, i.e.,

$$\tilde{u}_i = \overline{u}_i \quad on \quad S_u . \tag{A.36}$$

Theorem of Minimum Potential Energy. Assume that the mixed boundary value problem (A.17) has the solution $\boldsymbol{u}$. Then,

$$\Pi(\boldsymbol{u}) \le \Pi(\tilde{\boldsymbol{u}}) \tag{A.37}$$

for every kinematically admissible displacement $\tilde{\boldsymbol{u}}$.

Demonstration

Since $\boldsymbol{u}$ is a solution of the mixed boundary value problem (A.17) and $\tilde{\boldsymbol{u}}$ is a *kinematically admissible displacement*, the difference,

$$\delta\boldsymbol{u} = \tilde{\boldsymbol{u}} - \boldsymbol{u} \tag{A.38}$$

satisfies condition (A.32) so as to be a virtual displacement. Expressing $\tilde{\boldsymbol{u}} = \boldsymbol{u} + \delta\boldsymbol{u}$ the potential energy is,

$$\begin{aligned}\Pi(\tilde{u}_i) = \Pi(u_i + \delta u_i) &= \int_\Omega W(\varepsilon_{ij} + \delta\varepsilon_{ij})dv - \int_{\partial\Omega} \overline{t}_i(u_i + \delta u_i)ds - \int_\Omega f_i(u_i + \delta u_i)dv \\ &= \int_\Omega W(\varepsilon_{ij} + \delta\varepsilon_{ij})dv - \int_{\partial\Omega} \overline{t}_i \delta u_i ds - \int_{\partial\Omega} \overline{t}_i u_i ds - \int_\Omega f_i \delta u_i dv - \int_\Omega f_i u_i dv + \left(\int_\Omega W(\varepsilon_{ij})dv - \int_\Omega W(\varepsilon_{ij})dv\right)\end{aligned}$$

Note here the strain energy is added and subtracted at the end. Recalling (A.27d), we write the last expression in the following way,

$$\Pi(u_i + \delta u_i) - \Pi(u_i) = \int_\Omega \left(W(\varepsilon_{ij} + \delta\varepsilon_{ij}) - W(\varepsilon_{ij})\right)dv - \int_{\partial\Omega} \overline{t}_i \delta u_i ds - \int_\Omega f_i \delta u_i dv \tag{b}$$

Next, we expand the strain energy density in a power series and keep the terms up to the second order,

$$W(\varepsilon_{ij} + \delta\varepsilon_{ij}) = W(\varepsilon_{ij}) + \frac{\partial W(\varepsilon_{ij})}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{1}{2}\frac{\partial^2 W(\varepsilon_{ij})}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\delta\varepsilon_{ij}\delta\varepsilon_{kl}\,. \tag{c}$$

Inserting into (b) and recalling that $\partial W(\varepsilon_{ij}) / \partial\varepsilon_{ij} = \sigma_{ij}$ we obtain,

$$\Pi(u_i + \delta u_i) - \Pi(u_i) = \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}dv - \int_{\partial\Omega} \overline{t}_i \delta u_i ds - \int_\Omega f_i \delta u_i dv + \int_\Omega \frac{1}{2}\frac{\partial^2 W(\varepsilon_{ij})}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\delta\varepsilon_{ij}\delta\varepsilon_{kl}dv\,.$$

The first three terms cancel due to the principle of virtual work (A.34a) and thus,

$$\Pi(u_i + \delta u_i) - \Pi(u_i) = \int_\Omega \frac{1}{2}\frac{\partial^2 W(\varepsilon_{ij})}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\delta\varepsilon_{ij}\delta\varepsilon_{kl}dv\,. \tag{d}$$

Since $W(\varepsilon_{ij})$ is positive definite, the integrand in (d) is positive definite. Thus, (A.37) is demonstrated.

The theorem of *minimum potential energy* asserts that the difference between the global strain energy and the work done by the body forces and prescribed tractions takes a smaller value for the mixed boundary problem than any kinematically admissible displacement. Note that the standard finite element method for linear elasticity is based on this principle.

To apply the principle of minimum potential energy, the strain energy should be expressed in terms of strains or displacements and not containing any stress components. This is because of the definition of the strain energy $\delta W(\varepsilon_{ij})$. In several problems with beams, it is convenient to express the strain components in terms of a single deformation or deflection parameter. A typical example is the deflection $u_2(x_1)$ of a beam. In the strain energy density $W(\varepsilon_{ij})$ where the strains are explicit functions of $u_2(x_1)$ we have,

$$\delta W(\varepsilon_{ij}) = \left[\frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}} \frac{d\varepsilon_{ij}}{du_2}\right] du_2 \quad \Rightarrow \quad \delta W(\varepsilon_{ij}) = \frac{dW(\varepsilon_{ij})}{du_2} \delta u_2 .$$

Examples

Below we demonstrate examples using the energy methods discussed above. We also recall the theorems of Castigliano with representative applications.

1a. Finite changes and virtual changes in potential energy. The principle of virtual work or minimum potential energy is expressed as $\delta\Pi = 0$. From this principle, we can obtain the governing equations of the system. Let us discuss the difference between $\delta\Pi = 0$ and potential energy changes due to a finite change in displacement, i.e. $\Delta\Pi > 0$.

The potential energy takes its minimum value for a given system in equilibrium. If this minimum value is $\Pi(\boldsymbol{u})$ we know from calculus that $\Delta\Pi = \Pi(\boldsymbol{u}+\Delta\boldsymbol{u}) - \Pi(\boldsymbol{u})$ must be larger than zero when $\Delta\boldsymbol{u}$ $(\Delta u_1, \Delta u_2, \Delta u_3)$ is a small but finite quantity.

The variation $\delta\Pi = 0$ corresponds to a virtual displacement $\delta\boldsymbol{u}$ $(\delta u_1, \delta u_2, \delta u_3)$. From calculus, we know that a function $F(x)$ is stationary at x if $dF(x)/dx = 0$ or $dF(x) = 0$ for an infinitesimal change in the variable.

For an actual infinitesimal displacement, the actual changes in the forces due to the displacement are small and can be neglected in comparison to the forces themselves. Thus, a virtual displacement can be considered as an infinitesimal actual displacement, which allows us to write $\delta\Pi = 0$.

1b. Statically indeterminate truss: a truss *ABCD* with three members is subjected to a force *P* at D (Figure A2). After loading, point *D* is displaced at *D*'. Using the principle of virtual work and the compatibility of deformation at *D*, calculate the tensions in the three members of the truss. The geometry, Young's modulus *E* and cross section *S* are known.

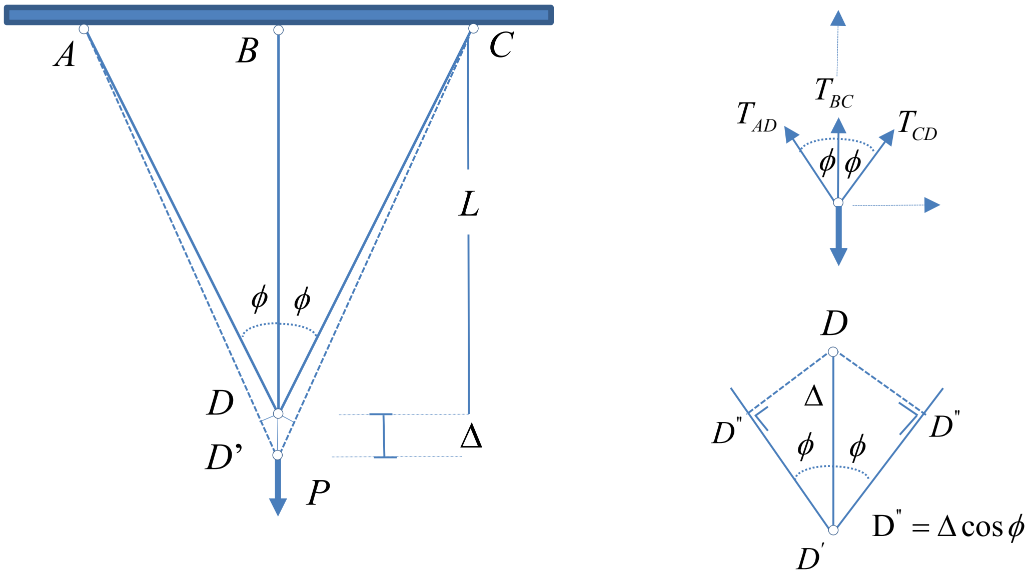

Fig. A2: A three-member truss subjected to a vertical force P at D. Also shown are the forces on each member and the displacement of point D.

The given structure is statically indetermined of degree one. To find the forces we should impose the compatibility of deformation at point *D* and determine the vertical displacement Δ.

Given the symmetry of the structure, the tensions in the bars are,

$$T_{BD} = \frac{SE}{L}\Delta, \qquad T_{AD} = T_{CD} = \frac{SE}{L/\cos\phi}(\Delta\cos\phi) = \frac{SE\cos^2\phi}{L}\Delta. \qquad \text{(a)}$$

Now, to determine Δ we should express the strain energy in terms of displacements and apply the virtual work principle. Thus, for each bar in tension (Del Pedro et al, 2012), the strain energies are given by,

$$U_{BD} = \frac{1}{2}T_{BD}\Delta = \frac{1}{2}T_{BD}\frac{T_{BD}L}{ES} = \frac{1}{2}\frac{T_{BD}^2 L}{ES} = \frac{1}{2}\frac{ES}{L}\Delta^2. \qquad \text{(b)}$$

Similarly,

$$U_{AD} = \frac{1}{2}\frac{ES}{L}\Delta^2 \cos^3\phi . \qquad \text{(c)}$$

We can now express the potential energy of the system and its variation as,

$$\Pi = U - W = \left(U_{BD} + 2U_{AD}\right) - P\Delta \;,\; \delta\left[\left(U_{BD} + 2U_{AD}\right) - P\Delta\right] = 0 \qquad \text{(d)}$$

from which we have successively,

$$\Rightarrow P = \frac{\delta\left(U_{BD} + 2U_{AD}\right)}{\delta\Delta} \qquad \text{(A.39)}$$

or

$$\delta\left(\frac{1}{2}\frac{ES}{L}\Delta^2 + 2\frac{1}{2}\frac{ES}{L}\Delta^2\cos^3\phi - P\Delta\right) = 0$$

$$\Rightarrow \frac{1}{2}\frac{ES}{L}(2\Delta\delta\Delta) + 2\frac{1}{2}\frac{ES}{L}(2\Delta\delta\Delta)\cos^3\phi = P\delta\Delta$$

$$\Rightarrow \quad \frac{ES}{L}\Delta + 2\frac{ES}{L}\Delta \;\cos^3\phi = P \quad \Rightarrow \quad \Delta = \frac{LP}{ES(1+2\cos^3\phi)}.$$

With this displacement, the forces in the bars are calculated using (a),

$$T_{BD} = \frac{P}{1+2\cos^3\phi}, \quad T_{AD} = T_{CD} = \frac{P\cos^2\phi}{1+2\cos^3\phi}.$$

Castiglinano's first theorem. The expression (A.39) in the preceding example is called *Castigliano's first theorem* which states that, to calculate a single force, i.e. P_i, we allow a virtual displacement $\delta\Delta$ that is continuous everywhere and vanishes at all points of loading except along the force P_i. In this virtual displacement, a virtual strain energy and external virtual work $P_i\delta\Delta_i$ are produced. According to the minimum potential energy principle we have,

$$\Pi = U - W = U - P_i\Delta_i \quad \text{and} \quad \delta\Pi = \delta(U - P_i\Delta_i) = 0 \;\; \Rightarrow \;\; P_i = \frac{\delta U}{\delta\Delta_i}. \qquad \text{(A.40)}$$

The theorem can be applied to linear or nonlinear structural response because linearity is not invoked in the demonstration.

2a. **Deflection of a wire.** For the (initially horizontal) elastic wire, subjected to a force at its middle as shown in Figure A3, calculate the relation between P and Δ using the first Castigliano theorem. The length L, modulus E and cross-section S are known.

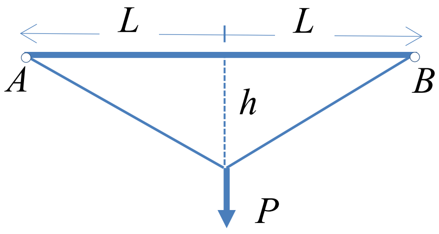

Fig. A3: Elastic wire subjected to a vertical force at its middle.

The elastic energy stored in the entire wire is given by,

$$U = 2\frac{ES}{2L}\Delta^2$$

where Δ is the elongation of half the initial length $2L$, which is expressed as,

$$\Delta = \left(L^2 + h^2\right)^{1/2} - L$$

Considering small deflections h, we expand the expression in series and retain the two first terms to obtain,

$$\Delta = L\left[1 + \frac{1}{2}\left(\frac{h}{L}\right)^2 + \ldots\right] - L \quad \Rightarrow \quad \Delta \simeq \frac{h^2}{2L}$$

Thus, the energy becomes,

$$U = \frac{ES}{4L^3}h^4.$$

From Castigilano's first theorem we have,

$$P = \frac{\partial U}{\partial h} = \frac{ES}{L^3}h^3$$

Note that the non-linearity comes from the geometry, i.e. large displacements, and not from the material.

Castigliano's second theorem. There is also the *Castigliano's second theorem,* which is very important in structural mechanics applications because it allows us to calculate displacements due to concentrated forces. This theorem states that in a linearly elastic structure, subjected to generalized forces $P_1, P_2, \ldots P_n$, the partial derivative of the strain energy with respect to a particular force gives the generalized displacement at the load point application along the direction of the force,

$$\Delta_i = \frac{\delta U(P_1, P_2, ..)}{\delta P_i}. \tag{A.41a}$$

Note here that the strain energy should be expressed in terms of the external loads and the theorem is applicable to linear elastic systems. The theorem can be easily demonstrated as follows.

The strain energy of an elastic system, subjected to $P_1, P_2, ...P_n$ in equilibrium is given by the theorem of Clapeyron (Del Pedro, et al, 2012),

$$U = \frac{1}{2} a_{ij} P_i P_j \quad (i,j = 1,2,...n) \tag{a}$$

where the constants $a_{ij} = a_{ji}$ are the influence coefficients that relate the deflection δ_k at point *k* along the force P_k due to the entire system of applied forces *n*,

$$\delta_k = a_{kq} P_q \qquad (k, q = 1,...n). \tag{b}$$

Next we take the derivative of the energy (a) with respect to the force P_k along which we seek the deflection δ_k,

$$\begin{aligned}
\frac{\partial U}{\partial P_k} &= \frac{1}{2} a_{ij} \frac{\partial P_i}{\partial P_k} P_j + \frac{1}{2} a_{ij} P_i \frac{\partial P_j}{\partial P_k} \\
&= \frac{1}{2} a_{ij} \delta_{ik} P_j + \frac{1}{2} a_{ij} P_i \delta_{jk} \\
&= \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{ik} P_i.
\end{aligned}$$

Due to the symmetry of the influence coefficients and that *j* and *i* are dummy indices, the last expression becomes,

$$\frac{\partial U}{\partial P_k} = \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{ik} P_i = \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{kj} P_j = a_{kj} P_j. \tag{c}$$

Due to (b) the last result is the deflection,

$$\delta_k = a_{ki} P_i. \qquad \text{(d)}$$

which is the deflection at the application point of the force P_i along its direction.

2b. Deflection and rotation due to a force. Calculate the vertical deflection and rotation at point A due to the force P of the structure in Figure A4, using Castigliano's second theorem and taking into account only the bending moment in each arm. The bending stiffness EI_3 and geometry are given.

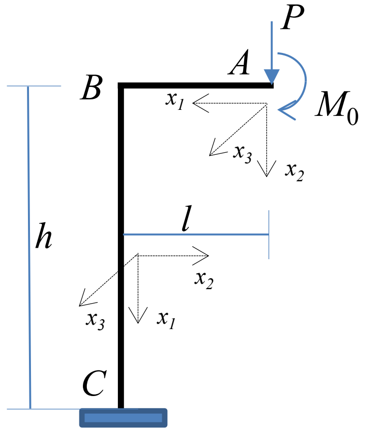

Fig. A4.

Here we will use (A.41a) to calculate the vertical deflection. For the rotation, we apply a fictitious moment M_0 at A, express the energy in terms of P, M_0 and take the derivative of the energy with respect to M_0 at $M_0 = 0$. For convenience, we express the moments in both arms of the structure due to both forces. Thus, the bending moments in each arm are,

$$0 \le x_1 \le l: \quad M(x_1) = Px_1 + M_0 \quad \Rightarrow \quad \frac{\partial M(x_1)}{\partial P} = x_1$$

$$0 \le x_1 \le h: \quad M(x_1) = Pl + M_0 \quad \Rightarrow \quad \frac{\partial M(x_1)}{\partial M_0} = 1$$

The vertical displacement, due to P, is given by (we set M_0=0)

$$\Delta = \frac{\partial U(P)}{\partial P} = \frac{1}{EI_3}\left[\int_0^l Px_1x_1dx_1 + \int_0^h Plldx_1\right] = \frac{Pl^3}{3EI_3} + \frac{Pl^2h}{EI_3}$$

and rotation by,

$$\theta = \frac{\partial U(P, M_0)}{\partial M_0} = \frac{1}{EI_3}\left[\int_0^l (Px_1 + M_0)\,dx_1 + \int_0^h (Pl + M_0)\,dx_1\right]_{M_0=0} = \frac{Pl^2}{2EI_3} + \frac{Plh}{EI_3}$$

Theorem of least work, or theorem of Menabrea. The derivative in (A.41a) to calculate the deflections along applied forces, is also used to determine unknown reactions in statically indetermined structures. In such cases, the strain energy is expressed in terms of the applied forces and all independent redundant forces, i.e. the unknown reactions $R_1, R_2, ...R_m$. These forces are obtained from the following equation,

$$\frac{\delta U(P_1, P_2, ..., R_1, R_2 ..., R_j ...)}{\delta R_j} = 0 \qquad (j = 1, 2..., m). \qquad \text{(A.41b)}$$

This last expression is referred to as the *theorem of least work*, or *theorem of Menabrea*. Note that the number of these equations is equal to the unknown redundant forces or reactions *m,* which is the necessary system of equations for the unknown reactions.

2c. calculation of reaction forces in statically indeterminate structure. For the beam loaded as shown in Figure A5, calculate the reaction forces.

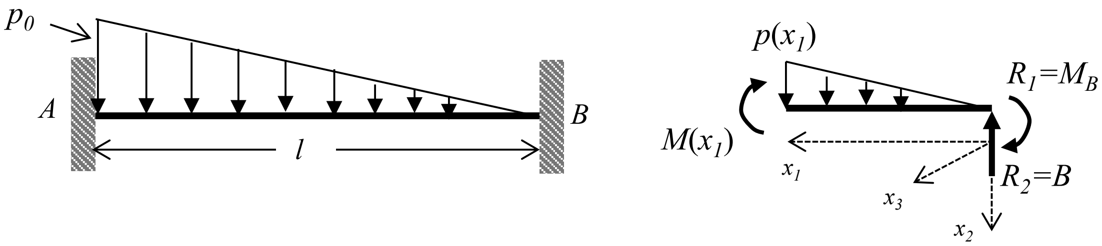

Fig. A5: Horizontal prismatic beam under linearly varying load.

The linearly distributed load at a distance x_1 from the left support is $p(x_1) = \frac{p_o}{\ell} x_1$. We will neglect the effects of the shear force and we assume that the horizontal forces are zero. The beam is statically indeterminate twice. Thus, we replace the reactions (force and moment in this example) on the right support by two generalized forces R_1, R_2. Next we express the bending moments and establish the derivatives of the energy with respect to R_1, R_2. The resulting two equations give the answer for the two forces unknown reactions,

$$-M(x_1) - R_1 + R_2 x_1 - \frac{1}{2} p(x_1) x_1 \frac{x_1}{3} = 0$$

$$\Rightarrow M(x_1) = -R_1 + R_2 x_1 - \frac{1}{6} p(x_1) x_1^{\,2} = -R_1 + R_2 x_1 - \frac{1}{6} \frac{p_o}{l} x_1^{\,3}$$

$$\frac{\partial M(x_1)}{\partial R_1} = -1; \qquad \frac{\partial M(x_1)}{\partial R_2} = x_1$$

$$\int_0^{\ell} \left(-R_1 + R_2 x_1 - \frac{1}{6} \frac{p_o}{l} x_1^3 \right) (-1) dx_1 = 0 \qquad \text{(a)}$$

$$\int_0^{l} \left(-R_1 + R_2 x_1 - \frac{1}{6} \frac{p_o}{l} x_1^3 \right) (x_1) dx_1 = 0 \qquad \text{(b)}$$

$$\left(R_1 x_1 - \frac{x_1^{\ 2}}{2} R_2 + \frac{p_o}{24} \frac{x_1^{\ 4}}{l} \right)_0^l = 0 \qquad \Rightarrow \quad R_1 - \frac{l}{2} R_2 + \frac{p_o l^2}{24} = 0$$

$$\left(-\frac{x_1^{\ 2}}{2} R_1 + \frac{x_1^{\ 3}}{3} R_2 - \frac{p_o}{30} \frac{x_1^{\ 5}}{l} \right)_0^l = 0 \qquad \Rightarrow -\frac{R_1}{2} + \frac{l}{3} R_2 - \frac{p_o l^2}{30} = 0$$

$$\Rightarrow R_1 = -\frac{p_o l^2}{30} \quad ; \qquad R_2 = -\frac{3 p_o l}{20}$$

With these forces known, the equilibrium equations of the entire beam give the reactions at B.

3. Uniformly loaded homogeneous string under lateral distributed load. A horizontal string in Figure A6, is loaded under a large tensile load P. A uniform transvers load q (N/m) is applied on the string. We assume that the application of q does not change the magnitude of P and that the string has neither resistance to bending nor weight.

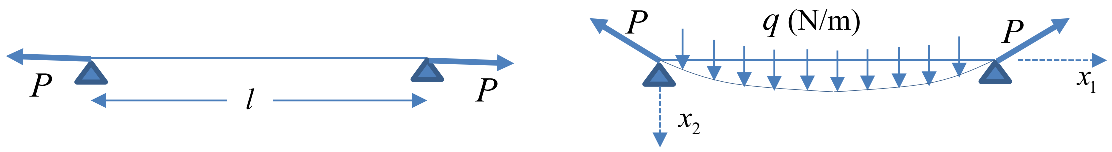

Fig. A6: A horizontal string under a force P (left) and an additional lateral distributed force on the right.

We further assume that the deflection $u_2(x_2)$ is small and the equation of equilibrium can be established in the unreformed configuration. We look for the governing equation of the string.

We consider the string on the left under tension and q = 0 as the reference state with zero potential energy. Upon loadings, the potential energy is,

$$\Pi = U - W \tag{a}$$

where,

$$W = \int_{\partial\Omega} t_i u_i ds = \int_0^l q u_2(x_1) dx_1 \ . \tag{b}$$

To evaluate U we need to determine the change in length of the string due to the lateral load q.

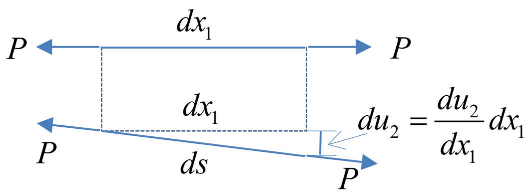

Fig. A7: A schematic of the deformed string showing parameters to calculate the deformation.

As seen in Figure A7, the elongation ds can be expressed as,

$$ds = \left(dx_1^2 + du_2^2\right)^{1/2} = dx_1\left(1 + (du_2/dx_1)^2\right)^{1/2} \cong dx_1\left(1 + \frac{1}{2}(du_2/dx_1)^2\right).$$

Thus, the infinitesimal change in length is $ds - dx_1$ results in the strain energy,

$$U = \int_0^l P(du_2 - dx_1) = \frac{P}{2}\int_0^l \left(\frac{du_2}{dx_1}\right)^2 dx_1 \qquad \text{(c)}$$

and the potential energy becomes,

$$\Pi = U - W = \frac{P}{2}\int_0^l \left(\frac{du_2}{dx_1}\right)^2 dx \; - \int_0^l q u_2 dx_1. \qquad \text{(d)}$$

The preceding analysis allows us to take the variation of the potential energy,

$$\delta\Pi = \delta(U - W) = \frac{2P}{2}\int_0^l \left(\frac{du_2}{dx_1}\right)\delta\left(\frac{du_2}{dx_1}\right)dx_1 - q\int_0^l \delta u_2 dx_1 \,. \qquad \text{(e)}$$

Since the operators d and δ may be interchanged (see A.33b) we obtain,

$$P\int_0^l \left(\frac{du_2}{dx_1}\right)\left(\frac{d\delta u_2}{dx_1}\right)dx_1 - q\int_0^l \delta u_2 dx_1 = 0 \qquad \text{(f)}$$

The first integral can be evaluated by parts,

$$P\int_0^l \left(\frac{du_2}{dx_1}\right)\left(\frac{d\delta u_2}{dx_1}\right)dx_1 = P\frac{du_2}{dx_1}\delta u_2\Big|_0^l - P\int_0^l \delta u_2 \frac{d^2u_2}{dx_1^{\,2}}dx_1 = -P\int_0^l \delta u_2 \frac{d^2u_2}{dx_1^{\,2}}dx_1$$

since $\delta u_2 = 0$ for $x_1 = 0,\ x_2 = l$.

Therefore,

$$\delta\Pi = -P\int_0^l \delta u_2 \frac{d^2u_2}{dx_1^{\ 2}}dx_1 - q\int_0^l \delta u_2 dx_1 = 0 \quad \Rightarrow \quad \int_0^l \left(P\frac{d^2u_2}{dx_1^{\ 2}} + q \right)\delta u_2 dx_1 \,.$$

And since δu_2 is arbitrary we find,

$$P\frac{d^2u_2}{dx_1^2} + q = 0 \qquad \text{(g)}$$

which is the equilibrium equation of the string in terms of the vertical deflection u_2.

4. Deflection of a beam subjected to a uniform load. Consider the prismatic beam in Figure A8. The section's geometry and elastic constants are known. Analyze the beam using the principle of minimum potential energy.

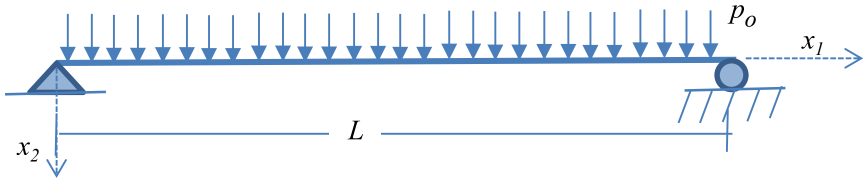

Fig. A8: Simply supported beam under uniformly distributed load.

From beam theory we have,

$$U = \int_0^L \frac{M^2(x_1)}{2EI_3}dx_1\,, \qquad \text{(a)}$$

$$M(x_1) = -EI_3\frac{d^2u_2}{dx_1^2} \qquad \text{(b)}$$

and thus, $U = \int_0^L \frac{M^2(x_1)}{2EI_3}dx_1 = EI_3\int_0^L \frac{1}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1$. (c)

The potential energy becomes,

$$\Pi = U - W = EI_3\int_0^L \frac{1}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 - \int_0^L p_0 u_2 dx_1 \qquad \text{(d)}$$

And the principle of minimum potential energy is expressed as,

$$\delta\Pi = \delta\left(U - \mathbb{W}\right) = EI_3\int_0^L \left(\frac{d^2u_2}{dx_1^2}\right)\frac{d^2(\delta u_2)}{dx_1^2}dx_1 - \int_0^L p_0\delta u_2 dx_1 = 0$$

The first integral is integrated by parts twice as shown in the following steps,

$$EI_3\int_0^L \left(\frac{d^2u_2}{dx_1^2}\right)\frac{d^2(\delta u_2)}{dx_1^2}dx_1 =$$

$$= EI_3\left[\frac{d^2u_2}{dx_1^2}\frac{d(\delta u_2)}{dx_1}\right]_0^L - EI_3\int_0^L \left(\frac{d(\delta u_2)}{dx_1}\right)d\left(\frac{d^2u_2}{dx_1^2}\right) = EI_3\left[\frac{d^2u_2}{dx_1^2}\frac{d(\delta u_2)}{dx_1}\right]_0^L - EI_3\int_0^L \left(\frac{d(\delta u_2)}{dx_1}\right)\left(\frac{d^3u_2}{dx_1^3}\right)dx_1$$

$$= EI_3\left[\frac{d^2u_2}{dx_1^2}\frac{d(\delta u_2)}{dx_1}\right]_0^L - EI_3\int_0^L \left(\frac{d^3u_2}{dx_1^3}\right)d\left(\delta u_2\right) = EI_3\left[\frac{d^2u_2}{dx_1^2}\frac{d(\delta u_2)}{dx_1} - \frac{d^3u_2}{dx_1^3}\delta u_2\right]_0^L + EI_3\int_0^L \left(\delta u_2\right)d\left(\frac{d^3u_2}{dx_1^3}\right)$$

$$= EI_3\left[\frac{d^2u_2}{dx_1^2}\frac{d(\delta u_2)}{dx_1} - \frac{d^3u_2}{dx_1^3}\delta u_2\right]_0^L + EI_3\int_0^L \delta u_2\left(\frac{d^4u_2}{dx_1^4}\right)dx_1.$$

Here the part in the brackets is zero at the two ends of the beam. Thus, combining the results with (d) we get,

$$\delta\Pi = \delta\left(U - \mathbb{W}\right) = EI_3\int_0^L \delta u_2\left(\frac{d^4u_2}{dx_1^4}\right)dx_1 - \int_0^L p_0\delta u_2 dx_1 = \int_0^L EI_3\left(\frac{d^4u_2}{dx_1^4} - p_0\right)\delta u_2 dx_1 = 0$$

Since δu_2 is arbitrary, the last equality is possible only when,

$$\frac{d^4u_2}{dx_1^4} - p_0 = 0$$

which is the governing equation of the beam in Figure A8. Note that, the theorem of the minimum potential energy for the loaded beam can be demonstrated following the procedure in the Example 3.

Rayleigh-Ritz Method. This method is very convenient to obtain approximate solutions using the principle of minimum potential energy. The essential elements of the method can be stated as follows: Firstly, a form of the deflection curve, containing a number of unknown parameters a_n $(n = 1, 2, ...)$ is assumed that satisfies the *geometric* boundary conditions. Another condition referred to as the *static* boundary condition which refers to applied forces and/or moments need not be fulfilled. Secondly, with the assumed solution we determine the potential energy of the system in terms of parameters $a_n \ (n=1,2,...)$. As the potential energy Π must be a minimum at equilibrium, we have,

$$\frac{\partial \Pi}{\partial a_1}=0, \ \frac{\partial \Pi}{\partial a_2}=0,..., \ \frac{\partial \Pi}{\partial a_n}=0$$

These algebraic equations are solved for $a_n \ (n=1,2,...)$ and are introduced in the assumed solution for the deflection curve to obtain the solution for the problem. In practice only a small number of parameters is necessary for an approximate solution. The accuracy of the approximation depends on how close the assumed shape of the deflection curve is to the exact shape.

5. Deflection of s simply supported beam. For the simply supported beam showing in Figure A8, determine the deflection $u_2(x_2)$ by employing a power series,

$$u_2(x_1)=a_1x_1(L-x_1)+a_2x_1^2(L-x_1)^2+.... \qquad \text{(a)}$$

For the beam we have *EI* = constant, *L* is given the distributed load is constant. Note that the form satisfies the end conditions of the beam. As an example, let's consider one term of the series,

$$u_2(x_1)=a_1x_1(L-x_1). \qquad \text{(b)}$$

The potential energy is given as,

$$\Pi=U-W=\int_0^L \frac{EI_3}{2}\left(\frac{d^2u_2(x_1)}{dx_1^2}\right)^2 dx-\int_0^L p_o u_2(x_1)dx_1 . \qquad \text{(c)}$$

Note that the energy of a beam under bending is given by the following expression (Del Pedro, et al, 2012),

$$U=\int_0^L \frac{M^2(x_1)}{2EI_3}dx_1 \ , \quad M(x_1)=-EI_3\frac{du_2^2(x_1)}{dx_1^2} \ \Rightarrow \ U=\int_0^L \frac{EI_3}{2}\left(\frac{d^2u_2(x_1)}{dx_1^2}\right)^2 dx .$$

Using (b) in (c) and integrating we obtain,

$$\Pi=U-W=\int_0^L \frac{EI_3}{2}(-2a_1)^2 dx_1-\int_0^L a_1p_ox_1(L-x_1)dx_1 .$$

Carrying out the integration and using the minimization, we obtain,

$$u_2(x_1) = \frac{p_o L^4}{4EI_3}\left(\frac{x_1}{L} - \left(\frac{x_1}{L}\right)^2\right). \qquad \text{(d)}$$

At mid-span expression (d) becomes,

$$u_2(L/2) = \frac{p_o L^4}{96EI}. \qquad \text{(e)}$$

This result differs from the exact solution, $u_2(L/2) = p_o L^4 / 76.8EI_3$, by 17%. By retaining the second term in (a), the method provides the exact solution at mid-span. Note that other approximations can be considered, i.e., a trigonometric series instead of (a).

6. Deflection of a statically indeterminate beam. For the beam shown in Figure A9, determine the deflection at mid-span. Use as a deflection curve the following polynomial,

$$u_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

where a_0, a_1, a_2, a_3 are constants.

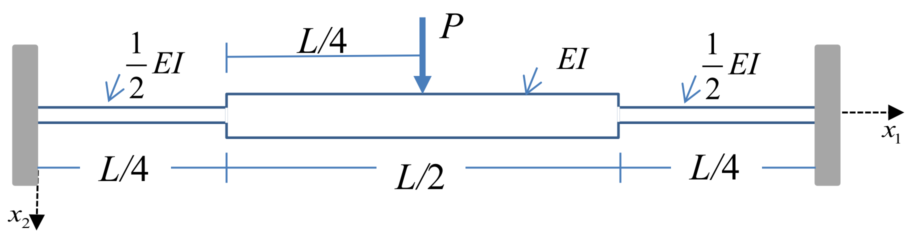

Fig. A9: A non-uniform beam with fixed ends.

The adopted solution should satisfy the geometric boundary conditions. Thus,

$$u_2(x_1)\Big|_{x_1=0} = 0 \quad (a); \qquad \frac{du_2(x_1)}{dx_1}\Big|_{x_1=0} = 0 \quad (b)$$

$$\frac{du_2(x_1)}{dx_1}\Big|_{x_1=L/2} = 0 \quad (c); \qquad u_2(x_1)\Big|_{x_1=L/2} = \Delta \quad (d)$$

In (d), Δ is the mid-span deflection to be evaluated. We apply these condition on the assumed deflection function.

From (a) and (b) we obtain, $a_1 = a_0 = 0$

From (c) we have $a_2 = -3a_3 L/4$

From (d) we have $a_3 = -16\Delta / L^3$.

With these parameters, the deflection curve takes the form,

$$u_2(x_1) = \frac{4\Delta x_1^2}{L^3}(3L - 4x_1) \qquad 0 \le x_1 \le L/2$$

and the strain energy becomes,

$$U = 2\left(\frac{EI_3/2}{2}\right)\int_o^{L/4}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 + 2\left(\frac{EI_3}{2}\right)\int_{L/4}^{L/2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 .$$

Introducing the deflection in the strain energy and integrating we obtain,

$$U = \frac{72EI_3\Delta^2}{L^3}.$$

The potential energy is,

$$\Pi = U - W = \frac{72EI_3\Delta^2}{L^3} - P\Delta .$$

The mid-span deflection is obtained by imposing the condition,

$$\frac{\partial \Pi}{\partial \Delta} = \frac{\partial}{\partial \Delta}\left(\frac{72EI_3\Delta^2}{L^3} - P\Delta\right) = 0 \qquad \Rightarrow \qquad \Delta = \frac{PL^3}{144EI_3}.$$

7a. Buckling of a vertical beam. Use the Rayleigh-Ritz method to determine the buckling load of a straight, uniform column subjected to a vertical compression load *P* (Figure A10). We assume that the deflection takes the form,

$$u_2(x_1) = \frac{ax_1^2}{L^2} \qquad \text{(a)}$$

Here *a* represents the deflection at the free end. Note that form (a) satisfies the geometric boundary condition at $x_1 = 0$.

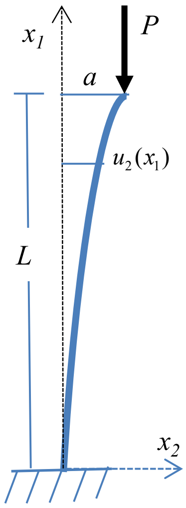

Fig. A10: Prismatic column under compression.

The potential energy of the column is,

$$\Pi = U - W = \int_0^L \frac{M^2(x_1)}{2EI_3} dx_1 - P\Delta . \quad \text{(b)}$$

Similar to the Example 5, l the strain energy is,

$$M(x_1) = -EI_3 \frac{du_2^2(x_1)}{dx_1^2} \quad \Rightarrow \quad U = \int_0^L \frac{EI_3}{2} \left(\frac{d^2u_2(x_1)}{dx_1^2} \right)^2 dx_1 \quad \text{(c)}$$

Regarding the potential energy of the applied load, the vertical displacement Δ is calculated by considering the change in length of the beam (see also Example A3),

$$ds = \left(dx_1^2 + du_2^2\right)^{1/2} = dx_1 \left(1 + (du_2 / dx_1)^2\right)^{1/2} \cong dx_1 \left(1 + \frac{1}{2}(du_2 / dx_1)^2\right)$$

$$d\Delta = ds - dx_1 = \frac{1}{2}(du_2 / dx_1)^2 dx_1$$

Thus,

$$W = P\Delta = \int_0^L P(ds - dx_1) = \frac{P}{2}\int_0^L \left(\frac{du_2}{dx_1}\right)^2 dx_1$$

$$\Pi = U - W = \int_0^L \frac{EI_3}{2} \left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 - \int_0^L \frac{P}{2}\left(\frac{du_2}{dx_1}\right)^2 dx_1 \quad \text{(d)}$$

Using (a) in (d) and integrating we obtain,

$$\Pi = \frac{2EI_3a^2}{L^3} - \frac{2Pa^2}{3L} . \quad \text{(e)}$$

Taking the derivative $\partial\Pi / \partial a = 0$ of (e) we get,

$$P_{cr} = 3\frac{EI_3}{L^2} . \quad \text{(f)}$$

This value is about 22% higher than the exact one, $P_{cr} = 2.4674 EI_3 / L^2$. We can do better if we express the energy in the beam in terms of the bending moment,

$$U = \int_0^L \frac{M^2(x_1)}{2EI_3} dx_1 \,. \tag{g}$$

The bending moment is given by $M(x_1) = P(a - u_2)$ and thus,

$$\Pi = U - \mathbb{W} = \int_0^L \frac{P^2(a-u_2)^2}{2EI_3} dx_1 - \int_0^L \frac{P}{2}\left(\frac{du_2}{dx_1}\right)^2 dx_1 \tag{h}$$

Using (a) in (h) the potential energy takes the form,

$$\Pi = \frac{8}{30}\frac{P^2 a^2 L}{EI_3} - \frac{2}{3}\frac{Pa^2}{L}$$

From the minimization $\partial \Pi / \partial a = 0$, we obtain the critical load, $P_{cr} = 2.50 \dfrac{EI_3}{L^2}$.

This value is 1.3% higher than the exact one. It is interesting to underline here the reason for a better estimation of the critical compressive force when the bending moment is used: When the deflection is used in calculating the energy, it is twice differentiated. This step 'reduces' the quality of approximation in the method because the error increases with differentiation. However, using the moment instead, we assume the second derivative of the deflection, given their relationship from beam theory, i.e. $M(x_1) \sim d^2u_2 / dx_1^2$ without subsequent differentiation.

7b. Buckling of a vertical bar. A tapered bar of constant thickness is loaded under compression as shown in Figure A11. Determine the critical load using the stationary value of the potential energy. The variation of the moment of inertia is,

$$\begin{aligned} I_3(x_1) &= I_0\left(1 + \frac{3x_1}{L}\right), \quad 0 \le x_1 \le \frac{L}{2} \\ I_3(x_1) &= I_0\left(4 - \frac{3x_1}{L}\right), \quad \frac{L}{2} \le x_1 \le L \end{aligned} \tag{a}$$

where I_0 is the moment of inertia at the ends of the bar. Use the principle of minimum potential energy and represent the deflection as,

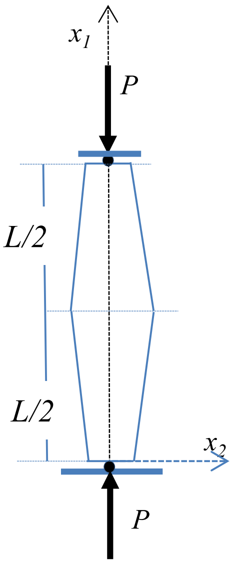

Fig. A11: Tapered bar under compression.

$$u_2(x_1) = a_1 \sin\frac{\pi x_1}{L} \tag{b}$$

The assumed form of the deflection satisfies the boundary conditions, i.e. $u_2(x_1) = 0$ at $x_1 = 0$ and $x_1 = L$.

The energy components are given by the expressions (d) in Example 7a,

$$\Pi = U - W = \int_0^L \frac{EI_3}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 - \int_0^L \frac{P}{2}\left(\frac{du_2}{dx_1}\right)^2 dx_1 \tag{c}$$

Due to symmetry, the energies are calculated by the integrals,

$$U = 2\int_0^{L/2} \frac{EI_0}{2}\left(1 + \frac{3x_1}{L}\right) a_1^2 \frac{\pi^4}{L^4} \sin^2\frac{\pi x_1}{L} dx_1 = \frac{8.215\pi^4 EI_0}{16L^3} a_1^2 \tag{d}$$

$$W = 2\int_0^{L/2} \frac{P}{2} a_1^2 \frac{\pi^2}{L^2} \cos^2\frac{\pi x_1}{L} dx_1 = \frac{\pi^2 P}{4L} a_1^2 \tag{e}$$

Thus,

$$\begin{aligned} &\Pi = U - W = \frac{8.215\pi^4 EI_0}{16L^3} a_1^2 - \frac{\pi^2 P}{4L} a_1^2 \\ &\delta\Pi = \delta(U - W) = \left(\frac{8.215\pi^4 EI_0}{16L^3} - \frac{\pi^2 P}{4L}\right)\delta a_1^2 = 0 \\ &\Rightarrow \frac{8.215\pi^4 EI_0}{16L^3} = \frac{\pi^2 P}{4L} \Rightarrow P_{cr} = 20.25\frac{EI_0}{L^2}. \end{aligned} \tag{f}$$

8. Applications to finite elements. In the finite element method, the solid is discretized by a finite number of elements connected at their nodes and along their inter-element boundaries as well.

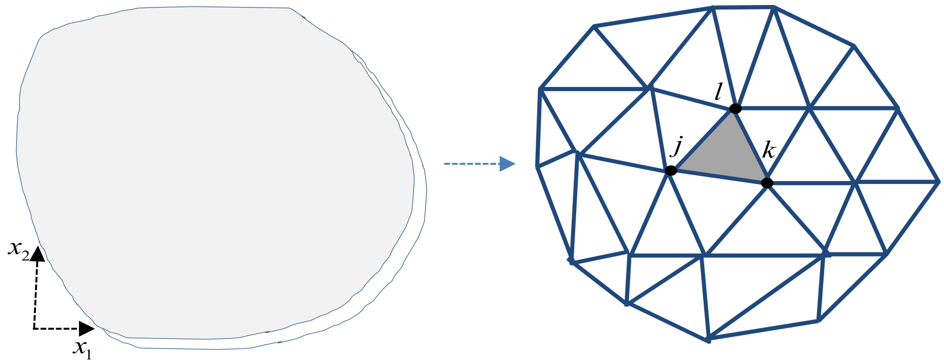

Fig. A12: Schematic of a discretization of a plate in plane stress and (e) *as an element with the three nodes.*

Here we must impose equilibrium nodal compatibility on the nodes and along the boundaries between the elements. The method relies on the minimization of the potential energy of the system expressed in terms of displacement functions. A typical illustrative example is shown in Figure A12 where a triangular element is used. Suppose that the plate is discretized by *m* elements. Using matrices and vectors, for each one, we define

- the nodal displacement vector,

$$\left(\tilde{\delta}\right)_e = \left(u_1^j, u_2^j, u_1^k, u_2^k, u_1^l, u_2^l\right) \qquad \text{(a)}$$

- the displacement functions within the element,

$$\left(\tilde{f}\right)_e = \left(u_i\,(x_1, x_2), u_2\,(x_1, x_2)\right) \qquad \text{(b)}$$

or,

$$\left(\tilde{f}\right)_e = [N]\left(\tilde{\delta}\right)_e \qquad \text{(c)}$$

The elements of $[N]$ are functions of position within the element.

- the strain displacement relation,

$$(\varepsilon)_e = [B]\left(\tilde{\delta}\right)_e \qquad \text{(d)}$$

- the constitutive response for each element,

$$(\sigma)_e = [C](\varepsilon)_e \qquad \text{(e)}$$

The principle of minimum potential energy is expressed for the entire body as follows (see

A.35),

$$\sum_1^m \int_{\Omega_e} \sigma_{ij}\delta\varepsilon_{ij}dv - \sum_1^m \int_{\partial\Omega_e} \overline{t}_i \delta u_i ds - \sum_1^m \int_{\Omega_e} f_i \delta u_i dv = 0 . \qquad \text{(f)}$$

Here $\Omega_e, \partial\Omega_e$ indicate the volume and surface of element *e*. Next, we use (b) in (f),

$$\sum_1^m \int_{\Omega_e} \left\{ (\delta\varepsilon)_e^T (\sigma)_e - \left(\delta\tilde{f}\right)_e^T (f)_e \right\} dv - \sum_1^m \int_{\partial\Omega_e} \left(\delta\tilde{f}\right)_e^T (t)_e ds = 0 \qquad \text{(g)}$$

Using (c), (d) and (e), the last expression becomes,

$$\sum_1^m \left(\delta\tilde{\delta}\right)_e^T \left[[k]_e \left(\tilde{\delta}\right)_e - (Q)_e \right] = 0 . \qquad \text{(h)}$$

In (h), $\left(\delta\tilde{\delta}\right)_e$ is the virtual nodal displacement vector of element *e,* $[k]_e$ is its stiffness matrix and $\left(\tilde{\delta}\right)_e, (Q)_e$ are vectors with the element's nodal displacements and nodal forces, due to body forces and surface tractions. Since the variations $\left(\delta\tilde{\delta}\right)_e$ are independent and arbitrary for a single element, (h) results in,

$$[k]_e \left(\tilde{\delta}\right)_e = (Q)_e . \qquad \text{(i)}$$

The sum in (g) can be expressed in the following matrix form for the entire discretized body,

$$\left(\delta\tilde{\delta}\right)^T \left[[K]\left(\tilde{\delta}\right) - (Q) \right] = 0 .$$

The last equation must be satisfied for arbitrary variations of all nodal displacements $\left(\delta\tilde{\delta}\right)$ and thus, we obtain the system of equations for the entire solid,

$$[K]\left(\tilde{\delta}\right) = (Q) \qquad \text{(j)}$$

where,

$$[K] = \sum_1^m [k]_e \text{ and } (Q) = \sum_1^m (Q)_e . \qquad \text{(k)}$$

The matrix $[K]$ and total force vector (Q) are identified by proper superposition of all elements stiffness and nodal forces.

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